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On Ostrowski Type Inequalities and Čebyšev Type Inequalities with Applications

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Abstract. In this paper, we obtain some new Ostrowski type inequalities and Čebyšev type inequalities for functions whose second derivatives absolute value are convex and second derivatives belongs to L_p spaces. Applications to a composite quadrature rule, to probability density functions, and to special means are also given.

1. Introduction

In 1938 Ostrowski obtained a bound for the absolute value of the difference of a function to its average over a finite interval. The theorem is well known in the literature as Ostrowski's integral inequality [25]:

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\left\| f' \right\|_\infty = \sup_{t \in (a,b)} \left| f'(t) \right| < \infty$. Then, the inequality holds:*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a) \left\| f' \right\|_\infty \tag{1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

In 1976, Milovanović and Pečarić proved a generalization of the Ostrowski inequality for n-times differentiable mappings (see for example [23, p.468]). In [15], [16], Dragomir and Wang extended the result (1) and applied the extended result to numerical quadrature rules and to the estimation of error bounds for some special means. Also, in [31], Sofo and Dragomir extended the result (1) in the L_p norm. In [10]-[12], Dragomir further extended the (1) to incorporate mappings of bounded variation, Lipschitzian and monotonic mappings. For recent results and generalizations conserning Ostrowski's integral inequality see [1]-[23], [31], [32], and the references therein.

In [7], Cerone and Dragomir find the following perturbed trapezoid inequalities:

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Theorem 1.2. *Let $f : [a,b] \rightarrow \mathbb{R}$ be such that the derivative f' is absolutely continuous on $[a,b]$. Then, the inequality holds:*

$$\left| \int_a^b f(t)dt - \frac{b-a}{2}\left[f(b)+f(a)\right] + \frac{(b-a)^2}{8}\left[f'(b)-f'(a)\right] \right| \tag{2}$$

$$\leq \begin{cases} \frac{(b-a)^3}{24}\left\|f''\right\|_\infty & \text{if } f'' \in L_\infty[a,b] \\ \frac{(b-a)^{2+\frac{1}{q}}}{8(2q+1)^{\frac{1}{q}}}\left\|f''\right\|_p & \text{if } f'' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \frac{(b-a)^2}{8}\left\|f''\right\|_1 & \text{if } f'' \in L_1[a,b] \end{cases}$$

for all $t \in [a,b]$.

In recent years a number of authors have considered an error analysis for some known and some new quadrature formulas. They used an approach from the inequalities point of view. For example, the midpoint quadrature rule is considered in [7], [18], [27], the trapezoid rule is considered in [7], [19], [33]. In most cases estimations of errors for these quadrature rules are obtained by means of derivatives and integrands.

In the literature, there are several classical and analytic inequalities for functions in related studies about inequalities. One of them was established by Čebyšev in [8] as following;

$$\left|T(f,g)\right| \leq \frac{1}{12}(b-a)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty, \tag{3}$$

where $f,g : [a,b] \rightarrow \mathbb{R}$ are absolutely continuous functions whose derivatives $f',g' \in L_\infty[a,b]$ and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{4}$$

which is called the Čebyšev functional, provided the integrals in (4) exist.

Several results related to the inequalities (3) and (4) can be found in the literature, see [17]-[13]. In [26], Pachpatte proved some inequalities similar to the inequality (3) as following;

Theorem 1.3. *Let $f,g : [a,b] \rightarrow \mathbb{R}$ be absolutely continuous functions whose derivatives $f',g' \in L_p[a,b]$, $p>1$. Then, we have the inequalities*

$$\left|T(f,g)\right| \leq \frac{1}{(b-a)^3}\left\|f'\right\|_p\left\|g'\right\|_p\int_a^b (B(x))^{\frac{2}{q}}dx,$$

$$\left|T(f,g)\right| \leq \frac{1}{2(b-a)^2}\int_a^b\left[\left|g(x)\right|\left\|f'\right\|_p + \left|f(x)\right|\left\|g'\right\|_p\right](B(x))^{\frac{1}{q}}dx,$$

where

$$B(x) = \frac{(x-a)^{q+1}+(b-x)^{q+1}}{q+1}$$

for $x \in [a,b]$, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\|f'\|_p = \left(\int_a^b |f'(t)|^p \, dt\right)^{\frac{1}{p}}.$$

In this article, we first derive a general integral identity for twice differentiable functions. Then, we apply this identity to obtain our results and using functions whose second derivatives in absolute value at certain powers are convex, we obtained new inequalities related to the Ostrowski's type inequality. Later, using this identity, we obtain several new generalizations for Čebyšev type inequalities involving functions whose second derivatives belong to L_p spaces. Finally, we gave some applications for a composite quadrature rule, probability density functions and special means of real numbers.

2. On Ostrowski Like Inequalities

In order to prove our results, we need the following lemma[see, [21]]:

Lemma 2.1. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differantiable mapping on* I° *where* $a, b \in I$ *wiyh* $a < b$, *such that* $f'' \in L_1[a,b]$. *Then, the following equality holds*

$$\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)} - \frac{1}{2}f(x) = \frac{1}{2(b-a)}\int_a^b p(x,t)f''(t)\,dt$$

where

$$p(x,t) = \begin{cases} (t-a)(t-x), & t \in [a,x] \\ (t-b)(t-x), & t \in (x,b] \end{cases}$$

for all $x \in [a,b]$.

Proof. A simple proof of the equality can be done by performing integration by parts. The details are left to the interested reader. □

Let us begin with the following Theorem.

Theorem 2.2. *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a twice differantiable mapping on* I° *where* $a, b \in I$ *with* $a < b$, *such that* $f'' \in L_1[a,b]$. *If* $|f''|$ *is convex on* $[a,b]$, *then the inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)} - \frac{1}{2}f(x)\right| \tag{5}$$

$$\leq \frac{(x-a)^3+(b-x)^3}{24(b-a)}\left|f''(x)\right| + \frac{(x-a)^3\left|f''(a)\right|+(b-x)^3\left|f''(b)\right|}{24(b-a)}$$

for all $x \in [a,b]$.

Proof. Using Lemma 2.1 and the modulus we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}-\frac{1}{2}f(x)\right| \\
\leq\;& \frac{1}{2(b-a)}\int_a^b |p(x,t)|\,|f''(t)|\,dt \\
=\;& \frac{1}{2(b-a)}\left\{\int_a^x (t-a)(x-t)\,|f''(t)|\,dt+\int_x^b (b-t)(t-x)\,|f''(t)|\,dt\right\}.
\end{aligned}$$

Since $|f''|$ is convex on $[a,b]=[a,x]\cup(x,b]$, therefore we have

$$|f''(t)|\leq\frac{t-a}{x-a}|f''(x)|+\frac{x-t}{x-a}|f''(a)|,\qquad t\in[a,x]$$

and

$$|f''(t)|\leq\frac{t-x}{b-x}|f''(b)|+\frac{b-t}{b-x}|f''(x)|,\qquad t\in(x,b],$$

which follows that,

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}-\frac{1}{2}f(x)\right| \\
\leq\;& \frac{1}{2(b-a)}\left\{\int_a^x (t-a)(x-t)\left[\frac{t-a}{x-a}|f''(x)|+\frac{x-t}{x-a}|f''(a)|\right]dt\right. \\
&\left.+\int_x^b (b-t)(t-x)\left[\frac{t-x}{b-x}|f''(b)|+\frac{b-t}{b-x}|f''(x)|\right]dt\right\} \\
=\;& \frac{(x-a)^3+(b-x)^3}{24(b-a)}|f''(x)|+\frac{(x-a)^3|f''(a)|+(b-x)^3|f''(b)|}{24(b-a)}
\end{aligned}$$

which completes the proof. □

Corollary 2.3. *In Theorem 2.2, if we choose*
i) $x=a$ or $x=b$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right|\leq\frac{(b-a)^2}{24}\left[|f''(a)|+|f''(b)|\right]. \tag{6}$$

ii) $x=\frac{a+b}{2}$, then we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right| \\
\leq\;& \frac{(b-a)^2}{192}\left(|f''(a)|+\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right).
\end{aligned} \tag{7}$$

Theorem 2.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differantiable mapping on I° where $a, b \in I$ wiyh $a < b$, such that $f'' \in L_1[a,b]$. If f'' is bounded, i.e., $\|f''\|_\infty = \sup_{x\in[a,b]} |f''(x)| < \infty$, then the inequality holds:*

$$\left| \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)} - \frac{1}{2}f(x) \right|$$

$$\leq \frac{\|f''\|_\infty}{12(b-a)}\left[(x-a)^3+(b-x)^3\right]$$

for all $x \in [a,b]$.

Proof. Using Lemma 2.1 and the modulus we have

$$\left| \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)} - \frac{1}{2}f(x) \right|$$

$$\leq \frac{1}{2(b-a)}\int_a^b |p(x,t)|\,|f''(t)|\,dt$$

$$\leq \frac{\|f''\|_\infty}{2(b-a)}\left\{\int_a^x (t-a)(x-t)dt + \int_x^b (b-t)(t-x)dt\right\}$$

$$= \frac{\|f''\|_\infty}{12(b-a)}\left[(x-a)^3+(b-x)^3\right]$$

which completes the proof. □

Corollary 2.5. *In Theorem 2.4, if we choose*

i) $x = a$ or $x = b$, then we have

$$\left| \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2} \right| \leq \frac{(b-a)^2}{12}\|f''\|_\infty .$$

ii) $x = \frac{a+b}{2}$, then we have

$$\left| \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] \right| \leq \frac{(b-a)^2}{48}\|f''\|_\infty .$$

Theorem 2.6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differantiable mapping on I° where $a, b \in I$ wiyh $a < b$, such that $f'' \in L_1[a,b]$. If $|f''|^q$, $q > 1$ is convex on $[a,b]$, then the inequality holds:*

$$\left| \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)} - \frac{1}{2}f(x) \right|$$

$$\leq \frac{B^{\frac{1}{p}}(p+1,p+1)}{2^{1+\frac{1}{q}}(b-a)}\left[(x-a)^3\left(|f''(x)|^q + |f''(a)|^q\right)^{\frac{1}{q}} + (b-x)^3\left(|f''(b)|^q + |f''(x)|^q\right)^{\frac{1}{q}}\right]$$

for all $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 2.1 and the Hölder's inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}-\frac{1}{2}f(x)\right| \\
\leq\ & \frac{1}{2(b-a)}\int_a^b |p(x,t)|\left|f''(t)\right|dt \\
=\ & \frac{1}{2(b-a)}\left\{\int_a^x (t-a)(x-t)\left|f''(t)\right|dt+\int_x^b (b-t)(t-x)\left|f''(t)\right|dt\right\} \\
\leq\ & \frac{1}{2(b-a)}\left(\int_a^x (t-a)^p(x-t)^p dt\right)^{\frac{1}{p}}\left(\int_a^x \left|f''(t)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{1}{2(b-a)}\left(\int_x^b (b-t)^p(t-x)^p dt\right)^{\frac{1}{p}}\left(\int_x^b \left|f''(t)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

Since $\left|f''\right|^q$ is convex on $[a,b]=[a,x]\cup(x,b]$, therefore we have

$$\left|f''(t)\right|^q\leq\frac{t-a}{x-a}\left|f''(x)\right|^q+\frac{x-t}{x-a}\left|f''(a)\right|^q,\qquad t\in[a,x]$$

and

$$\left|f''(t)\right|^q\leq\frac{t-x}{b-x}\left|f''(b)\right|^q+\frac{b-t}{b-x}\left|f''(x)\right|^q,\qquad t\in(x,b],$$

which follows that,

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}-\frac{1}{2}f(x)\right| \\
\leq\ & \frac{1}{2(b-a)}\left[(x-a)^{2p+1}B(p+1,p+1)\right]^{\frac{1}{p}}\left(\int_a^x\left[\frac{t-a}{x-a}\left|f''(x)\right|^q+\frac{x-t}{x-a}\left|f''(a)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+\frac{1}{2(b-a)}\left[(b-x)^{2p+1}B(p+1,p+1)\right]^{\frac{1}{p}}\left(\int_x^b\left[\frac{t-x}{b-x}\left|f''(b)\right|^q+\frac{b-t}{b-x}\left|f''(x)\right|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{(x-a)^3B^{\frac{1}{p}}(p+1,p+1)}{2^{1+\frac{1}{q}}(b-a)}\left(\left|f''(x)\right|^q+\left|f''(a)\right|^q\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^3B^{\frac{1}{p}}(p+1,p+1)}{2^{1+\frac{1}{q}}(b-a)}\left(\left|f''(b)\right|^q+\left|f''(x)\right|^q\right)^{\frac{1}{q}}
\end{aligned}$$

since $\frac{1}{p}+\frac{1}{q}=1,\ q>1$, which completes the proof. □

Corollary 2.7. *In Theorem 2.6, if we choose,*

i) $x=a$ or $x=b$, then we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right|\le\frac{B^{\frac{1}{p}}(p+1,p+1)(b-a)^2}{2^{1+\frac{1}{q}}}\left(\left|f''(a)\right|^q+\left|f''(b)\right|^q\right)^{\frac{1}{q}}. \tag{8}$$

ii) $x=\dfrac{a+b}{2}$, then we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right| \qquad (9)\\ \le\;& \frac{B^{\frac{1}{p}}(p+1,p+1)(b-a)^2}{2^{4+\frac{1}{q}}}\\ &\times\left[\left(\left|f''(a)\right|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q+\left|f''(b)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Theorem 2.8. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differantiable mapping on I° where $a,b\in I$ wiyh $a<b$, such that $f''\in L_1[a,b]$. If $\left|f''\right|^q$, $q\ge 1$ is convex on $[a,b]$, then the inequality holds:*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}-\frac{1}{2}f(x)\right|\\ \le\;& \frac{1}{12(b-a)}\left[(x-a)^3\left(\frac{\left|f''(x)\right|^q+\left|f''(a)\right|^q}{2}\right)^{\frac{1}{q}}+(b-x)^3\left(\frac{\left|f''(b)\right|^q+\left|f''(x)\right|^q}{2}\right)^{\frac{1}{q}}\right]\end{aligned}$$

for all $x\in[a,b]$, where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1 and the Hölder's inequality, we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}-\frac{1}{2}f(x)\right|\\ \le\;& \frac{1}{2(b-a)}\int_a^b \left|p(x,t)\right|\left|f''(t)\right|dt\\ =\;& \frac{1}{2(b-a)}\left\{\int_a^x (t-a)(x-t)\left|f''(t)\right|dt+\int_x^b (b-t)(t-x)\left|f''(t)\right|dt\right\}\\ \le\;& \frac{1}{2(b-a)}\left(\int_a^x (t-a)(x-t)dt\right)^{\frac{1}{p}}\left(\int_a^x (t-a)(x-t)\left|f''(t)\right|^q dt\right)^{\frac{1}{q}}\\ &+\frac{1}{2(b-a)}\left(\int_x^b (b-t)(t-x)dt\right)^{\frac{1}{p}}\left(\int_x^b (b-t)(t-x)\left|f''(t)\right|^q dt\right)^{\frac{1}{q}}\end{aligned}$$

Since $\left|f''\right|^q$ is convex on $[a,b]=[a,x]\cup(x,b]$, therefore we have

$$\left|f''(t)\right|^q \le \frac{t-a}{x-a}\left|f''(x)\right|^q+\frac{x-t}{x-a}\left|f''(a)\right|^q, \qquad t\in[a,x]$$

and

$$\left|f''(t)\right|^q \le \frac{t-x}{b-x}\left|f''(b)\right|^q+\frac{b-t}{b-x}\left|f''(x)\right|^q, \qquad t\in(x,b],$$

which follows that,

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}-\frac{1}{2}f(x)\right| \\
\le\ & \frac{1}{2(b-a)}\left[\frac{(x-a)^3}{6}\right]^{\frac{1}{p}}\left(\int_a^x (t-a)(x-t)\left[\frac{t-a}{x-a}\left|f''(x)\right|^q+\frac{x-t}{x-a}\left|f''(a)\right|^q\right]dt\right)^{\frac{1}{q}} \\
&+\frac{1}{2(b-a)}\left[\frac{(b-x)^3}{6}\right]^{\frac{1}{p}}\left(\int_x^b (b-t)(t-x)\left[\frac{t-x}{b-x}\left|f''(b)\right|^q+\frac{b-t}{b-x}\left|f''(x)\right|^q\right]dt\right)^{\frac{1}{q}} \\
=\ & \frac{1}{2(b-a)}\left[\frac{(x-a)^3}{6}\right]^{\frac{1}{p}}\left[\frac{(x-a)^3}{6}\right]^{\frac{1}{q}}\left(\frac{\left|f''(x)\right|^q+\left|f''(a)\right|^q}{2}\right)^{\frac{1}{q}} \\
&+\frac{1}{2(b-a)}\left[\frac{(b-x)^3}{6}\right]^{\frac{1}{p}}\left[\frac{(b-x)^3}{6}\right]^{\frac{1}{q}}\left(\frac{\left|f''(b)\right|^q+\left|f''(x)\right|^q}{2}\right)^{\frac{1}{q}}
\end{aligned}$$

which completes the proof. □

Corollary 2.9. *With the assumptions Theorem 2.8, if we take $q=1$, then we get*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}-\frac{1}{2}f(x)\right| \\
\le\ & \frac{1}{12(b-a)}\left[(x-a)^3\left(\frac{\left|f''(x)\right|+\left|f''(a)\right|}{2}\right)+(b-x)^3\left(\frac{\left|f''(b)\right|+\left|f''(x)\right|}{2}\right)\right]
\end{aligned}$$

for all $x\in[a,b]$.

Corollary 2.10. *In Corollary 2.9, if we choose*
i) $x=a$ or $x=b$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right| \le \frac{(b-a)^2}{24}\left[\left|f''(b)\right|+\left|f''(a)\right|\right].$$

ii) $x = \frac{a+b}{2}$, *then we have*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\right| \\ \leq \; & \frac{(b-a)^2}{96}\left[\left| f''\left(\frac{a+b}{2}\right)\right| + \frac{|f''(a)| + |f''(b)|}{2}\right]. \end{aligned}$$

The following result holds for concave functions:

Theorem 2.11. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differantiable mapping on I° where $a, b \in I$ wiyh $a < b$, such that $f'' \in L_1[a,b]$. If $|f''|^q$, $q \geq 1$ is concave on $[a,b]$, then the inequality holds:*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)\,dt - \frac{(x-a)f(a) + (b-x)f(b)}{2(b-a)} - \frac{1}{2}f(x)\right| \\ \leq \; & \frac{1}{12(b-a)}\left[(x-a)^3 \left| f''\left(\frac{x+a}{2}\right)\right| + (b-x)^3 \left| f''\left(\frac{b+x}{2}\right)\right|\right] \end{aligned}$$

for all $x \in [a,b]$.

Proof. Since $|f''|^q$ for $q \geq 1$ is concave, hence by power mean inequality, we have

$$|f''(\lambda x + (1-\lambda)y)|^q \geq \left(\lambda |f''(x)| + (1-\lambda)|f''(y)|\right)^q.$$

This implies that

$$|f''(\lambda x + (1-\lambda)y)| \geq \left(\lambda |f''(x)| + (1-\lambda)|f''(y)|\right),$$

and hence $\left|f''\right|$ is also concave. Therefore, from Lemma 2.1 and the Jensen's integral inequality, we have

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}-\frac{1}{2}f(x)\right| \\
&= \frac{1}{2(b-a)}\left\{\int_a^x (t-a)(x-t)\left|f''(t)\right|dt+\int_x^b (b-t)(t-x)\left|f''(t)\right|dt\right\} \\
&\leq \frac{1}{2(b-a)}\left\{\left(\int_a^x (t-a)(x-t)dt\right)\left|f''\left(\frac{\int_a^x t(t-a)(x-t)dt}{\int_a^x (t-a)(x-t)dt}\right)\right|\right. \\
&\quad \left.+\frac{1}{2(b-a)}\left(\int_x^b (b-t)(t-x)dt\right)\left|f''\left(\frac{\int_x^b t(b-t)(t-x)dt}{\int_x^b (b-t)(t-x)dt}\right)\right|\right\} \\
&= \frac{1}{2(b-a)}\left\{\frac{(x-a)^3}{6}\left|f''\left(\frac{\frac{(x-a)^3(x+a)}{12}}{\frac{(x-a)^3}{6}}\right)\right|+\frac{(b-x)^3}{6}\left|f''\left(\frac{\frac{(b-x)^3(x+b)}{12}}{\frac{(b-x)^3}{6}}\right)\right|\right\} \\
&= \frac{1}{12(b-a)}\left\{(x-a)^3\left|f''\left(\frac{x+a}{2}\right)\right|+(b-x)^3\left|f''\left(\frac{x+b}{2}\right)\right|\right\}
\end{aligned}$$

for all $x\in[a,b]$, which completes the proof. □

Corollary 2.12. *In Theorem 2.11, if we choose*

i) $x=a$ *or* $x=b$, *then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right|\leq\frac{(b-a)^2}{12}\left|f''\left(\frac{a+b}{2}\right)\right|$$

ii) $x=\frac{a+b}{2}$, *then we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{(b-a)}{96}\left[\left|f''\left(\frac{3a+b}{2}\right)\right|+\left|f''\left(\frac{a+3a}{2}\right)\right|\right].
\end{aligned}$$

3. On Čebyšev Type Inequalities

The following Čebyšev like ineqaulites hold:

Theorem 3.1. *Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions whose derivatives $f', g' \in L_p[a, b]$, $p > 1$. Then, we have the inequalities*

$$\begin{aligned}
&\left|\frac{1}{(b-a)}\int_a^b f(x)\,g(x)\,dx - \frac{M(a,b)}{3} - \frac{N(a,b)}{6}\right| \\
\leq\ & \frac{B^{\frac{1}{q}}(q+1,q+1)\left\|g''\right\|_p}{(b-a)^3}\int_a^b \left[\left[(x-a)\left|f(a)\right| + (b-x)\left|f(b)\right|\right][S(x)]^{\frac{1}{q}}\right]dx \\
&+\frac{B^{\frac{1}{q}}(q+1,q+1)\left\|f''\right\|_p}{(b-a)^3}\int_a^b \left[\left[(x-a)\left|g(a)\right| + (b-x)\left|g(b)\right|\right][S(x)]^{\frac{1}{q}}\right]dx \\
&+\frac{B^{\frac{2}{q}}(q+1,q+1)\left\|f''\right\|_p\left\|g''\right\|_p}{(b-a)^3}\int_a^b [S(x)]^{\frac{2}{q}}\,dx.
\end{aligned}$$

and

$$\begin{aligned}
&\left|T(f,g)\right| \leq \frac{1}{(b-a)}\int_a^b \left|f(x)\right|\left|g(x)\right|dx \\
&+\frac{1}{2(b-a)^2}\int_a^b \left[(x-a)\,N_1(a,x) + (b-x)\,N_2(b,x)\right]dx \\
&+\frac{B^{\frac{1}{q}}(q+1,q+1)}{2(b-a)^2}\int_a^b [S(x)]^{\frac{1}{q}}\left(\left|g(x)\right|\left\|f''\right\|_p + \left|f(x)\right|\left\|g''\right\|_p\right)dx
\end{aligned} \tag{10}$$

where B is beta function,

$$\begin{aligned}
S(x) &= (x-a)^{2q+1} + (b-x)^{2q+1} \\
M(a,b) &= f(a)g(a) + f(b)g(b) \\
N(a,b) &= f(a)g(b) + f(b)g(a) \\
N_1(a,x) &= \left|f(a)\right|\left|g(x)\right| + \left|g(a)\right|\left|f(x)\right| \\
N_2(b,x) &= \left|f(b)\right|\left|g(x)\right| + \left|g(b)\right|\left|f(x)\right|
\end{aligned}$$

for $x \in [a, b]$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1, we have the following identities,

$$\frac{1}{2}f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt = -\frac{(x-a)\,f(a) + (b-x)\,f(b)}{2(b-a)} - \frac{1}{2(b-a)}\int_a^b p(x,t)f''(t)\,dt \tag{11}$$

and

$$\frac{1}{2}g(x) - \frac{1}{b-a}\int_a^b g(t)\,dt = -\frac{(x-a)\,g(a) + (b-x)\,g(b)}{2(b-a)} - \frac{1}{2(b-a)}\int_a^b p(x,t)g''(t)\,dt. \tag{12}$$

By multiplying both sides of the inequalities (11) and (12), we have

$$
\begin{aligned}
&\frac{1}{4}f(x)\,g(x)-\frac{1}{2}f(x)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)\\
&-\frac{1}{2}g(x)\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)+\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right)\\
=\;&\frac{[(x-a)\,f(a)+(b-x)\,f(b)]\,[(x-a)\,g(a)+(b-x)\,g(b)]}{4\,(b-a)^2}\\
&+\frac{[(x-a)\,f(a)+(b-x)\,f(b)]}{4\,(b-a)^2}\left(\int_a^b p(x,t)g''(t)\,dt\right)\\
&+\frac{[(x-a)\,g(a)+(b-x)\,g(b)]}{4\,(b-a)^2}\left(\int_a^b p(x,t)f''(t)\,dt\right)\\
&+\frac{1}{4\,(b-a)^2}\left(\int_a^b p(x,t)f''(t)\,dt\right)\left(\int_a^b p(x,t)g''(t)\,dt\right).
\end{aligned}
$$

Integrating both sides of the above equality with respect to x over $[a,b]$ and dividing both sides of the result equality by $(b-a)$, we get

$$
\begin{aligned}
&\frac{1}{4\,(b-a)}\int_a^b f(x)\,g(x)\,dx\\
&-\frac{1}{4\,(b-a)^3}\int_a^b [(x-a)\,f(a)+(b-x)\,f(b)]\,[(x-a)\,g(a)+(b-x)\,g(b)]\,dx\\
=\;&\frac{1}{4\,(b-a)^3}\int_a^b\left[[(x-a)\,f(a)+(b-x)\,f(b)]\left(\int_a^b p(x,t)g''(t)\,dt\right)\right]dx\\
&+\frac{1}{4\,(b-a)^3}\int_a^b\left[[(x-a)\,g(a)+(b-x)\,g(b)]\left(\int_a^b p(x,t)f''(t)\,dt\right)\right]dx\\
&+\frac{1}{4\,(b-a)^3}\int_a^b\left(\int_a^b p(x,t)f''(t)\,dt\right)\left(\int_a^b p(x,t)g''(t)\,dt\right)dx.
\end{aligned}
$$

By using the properties of modulus and Hölder's integral inequality, we can write

$$\begin{aligned}
&\left|\frac{1}{4(b-a)}\int_a^b f(x)\,g(x)\,dx\right.\\
&\left.-\frac{1}{4(b-a)^3}\int_a^b \left[(x-a)f(a)+(b-x)f(b)\right]\left[(x-a)g(a)+(b-x)g(b)\right]dx\right|\\
\leq\ & \frac{1}{4(b-a)^3}\int_a^b \left[\left[(x-a)f(a)+(b-x)f(b)\right]\left(\int_a^b \left|p(x,t)\right|^q dt\right)^{\frac{1}{q}}\left(\int_a^b \left|g''(t)\right|^p dt\right)^{\frac{1}{p}}\right]dx\\
&+\frac{1}{4(b-a)^3}\int_a^b \left[\left[(x-a)g(a)+(b-x)g(b)\right]\left(\int_a^b \left|p(x,t)\right|^q dt\right)^{\frac{1}{q}}\left(\int_a^b \left|f''(t)\right|^p dt\right)^{\frac{1}{p}}\right]dx\\
&+\frac{1}{4(b-a)^3}\int_a^b \left[\left(\int_a^b \left|p(x,t)\right|^q dt\right)^{\frac{1}{q}}\left(\int_a^b \left|f''(t)\right|^p dt\right)^{\frac{1}{p}}\right]\left[\left(\int_a^b \left|p(x,t)\right|^q dt\right)^{\frac{1}{q}}\left(\int_a^b \left|g''(t)\right|^p dt\right)^{\frac{1}{p}}\right]dx.
\end{aligned}$$

That is

$$\begin{aligned}
&\left|\frac{1}{4(b-a)}\int_a^b f(x)\,g(x)\,dx-\frac{M(a,b)}{12}-\frac{N(a,b)}{24}\right|\\
\leq\ & \frac{B^{\frac{1}{q}}(q+1,q+1)\left\|g''\right\|_p}{4(b-a)^3}\int_a^b \left[\left[(x-a)\left|f(a)\right|+(b-x)\left|f(b)\right|\right]\left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{\frac{1}{q}}\right]dx\\
&+\frac{B^{\frac{1}{q}}(q+1,q+1)\left\|f''\right\|_p}{4(b-a)^3}\int_a^b \left[\left[(x-a)\left|g(a)\right|+(b-x)\left|g(b)\right|\right]\left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{\frac{1}{q}}\right]dx\\
&+\frac{B^{\frac{2}{q}}(q+1,q+1)\left\|f''\right\|_p\left\|g''\right\|_p}{4(b-a)^3}\int_a^b \left[(x-a)^{2q+1}+(b-x)^{2q+1}\right]^{\frac{2}{q}}dx.
\end{aligned}$$

For the proof of the inequality (10), if we multiply both sides of (11) and (12) by $g(x)$ and $f(x)$, respectively, we have

$$\frac{1}{2}f(x)g(x)-\frac{g(x)}{b-a}\int_a^b f(t)\,dt=-\frac{(x-a)f(a)+(b-x)f(b)}{2(b-a)}g(x)-\frac{g(x)}{2(b-a)}\int_a^b p(x,t)f''(t)\,dt \tag{13}$$

and

$$\frac{1}{2}f(x)\,g(x)-\frac{f(x)}{b-a}\int_a^b g(t)\,dt=-\frac{(x-a)g(a)+(b-x)g(b)}{2(b-a)}f(x)-\frac{f(x)}{2(b-a)}\int_a^b p(x,t)g''(t)\,dt \tag{14}$$

By adding the equalities (13) and (14), integrating both sides of the result equality with respect to x over $[a,b]$ and dividing both sides of the result equality by $(b-a)$, we get

$$
\begin{aligned}
& T(f,g) \\
= \; & \frac{1}{(b-a)}\int_a^b f(x)g(x)dx - \frac{1}{2(b-a)^2}\int_a^b \left[(x-a)f(a)+(b-x)f(b)\right]g(x)\,dx \\
& -\frac{1}{2(b-a)^2}\left[\int_a^b g(x)\left(\int_a^b p(x,t)f''(t)\,dt\right)dx\right] \\
& -\frac{1}{2(b-a)^2}\int_a^b \left[(x-a)g(a)+(b-x)g(b)\right]f(x)\,dx \\
& -\frac{1}{2(b-a)^2}\left[\int_a^b f(x)\left(\int_a^b p(x,t)g''(t)\,dt\right)dx\right].
\end{aligned}
$$

By using the properties of modulus and Hölder's integral inequality, we can write

$$
\begin{aligned}
& \left|T(f,g)\right| \\
\leq \; & \frac{1}{(b-a)}\int_a^b \left|f(x)\right|\left|g(x)\right|dx + \frac{1}{2(b-a)^2}\int_a^b \left[(x-a)\left|f(a)\right|+(b-x)\left|f(b)\right|\right]\left|g(x)\right|dx \\
& +\frac{1}{2(b-a)^2}\left[\int_a^b \left|g(x)\right|\left\{\left(\int_a^b \left|p(x,t)\right|^q dt\right)^{\frac{1}{q}}\left(\int_a^b \left|f''(t)\right|^p dt\right)^{\frac{1}{p}}\right\}dx\right] \\
& +\frac{1}{2(b-a)^2}\int_a^b \left[(x-a)\left|g(a)\right|+(b-x)\left|g(b)\right|\right]\left|f(x)\right|dx \\
& +\frac{1}{2(b-a)^2}\left[\int_a^b \left|f(x)\right|\left\{\left(\int_a^b \left|p(x,t)\right|^q dt\right)^{\frac{1}{q}}\left(\int_a^b \left|g''(t)\right|^p dt\right)^{\frac{1}{p}}\right\}dx\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(b-a)}\int_a^b \left|f(x)\right|\left|g(x)\right|dx + \frac{1}{2(b-a)^2}\int_a^b \left[(x-a)\left|f(a)\right| + (b-x)\left|f(b)\right|\right]\left|g(x)\right|dx \\
&\quad + \frac{1}{2(b-a)^2}\int_a^b \left[(x-a)\left|g(a)\right| + (b-x)\left|g(b)\right|\right]\left|f(x)\right|dx \\
&\quad + \frac{B^{\frac{1}{q}}(q+1,q+1)\left\|f''\right\|_p}{2(b-a)^2}\left[\int_a^b \left[(x-a)^{2q+1} + (b-x)^{2q+1}\right]^{\frac{1}{q}}\left|g(x)\right|dx\right] \\
&\quad + \frac{B^{\frac{1}{q}}(q+1,q+1)\left\|g''\right\|_p}{2(b-a)^2}\left[\int_a^b \left[(x-a)^{2q+1} + (b-x)^{2q+1}\right]^{\frac{1}{q}}\left|f(x)\right|dx\right]
\end{aligned}
$$

which completes the proof. □

4. Application to Probability Density Functions

Now, let X be a random variable taking values in the finite interval $[a,b]$ with the probability density function $f:[a,b]\to[0,1]$ and with the cumulative distribution function

$$F(x) = \Pr(X \le x) = \int_a^x f(t)dt.$$

The following result holds:

Proposition 4.1. *With the assumptions Theorem 2.2, we have the following inequality*

$$
\begin{aligned}
&\left|\frac{b-E(X)}{b-a} - \frac{(b-x)F(b)}{2(b-a)} - \frac{1}{2}F(x)\right| \qquad (15)\\
&\le \frac{(x-a)^3 + (b-x)^3}{24(b-a)}\left|f(x)\right| + \frac{(x-a)^3\left|f'(a)\right| + (b-x)^3\left|f'(b)\right|}{24(b-a)}
\end{aligned}
$$

for all $x \in [a,b]$, where $E(X)$ is the expectation of X.

Proof. Follows by (5) on choosing $f = F$ and taking into account

$$E(X) = \int_a^b t dF(t) = b - \int_a^b F(t)dt,$$

we obtain (15). □

Corollary 4.2. *In Proposition 4.1, if we choose*
i) $x = b$, then we get

$$\left|\frac{b-E(X)}{b-a} - \frac{1}{2}F(b)\right| \le \frac{(b-a)^2}{12}\left(\frac{\left|f'(a)\right| + \left|f(b)\right|}{2}\right)$$

ii) $x = a$, then we get

$$\left|\frac{b-E(X)}{b-a}-\frac{F(a)+F(b)}{2}\right| \leq \frac{(b-a)^2}{12}\left(\frac{\left|f'(a)\right|+\left|f(b)\right|}{2}\right)$$

iii) $x = \frac{a+b}{2}$, then we get

$$\left|\frac{b-E(X)}{b-a}-\frac{F(b)}{4}-\frac{1}{2}F\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{96}\left[\left|f'(a)\right|+\left|f\left(\frac{a+b}{2}\right)\right|+\left|f'(b)\right|\right].$$

5. Applications for Special Means

Recall the following means:
(a) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \; a,b \geq 0;$$

(b) The geometric mean

$$G = G(a,b) := \sqrt{ab}, \; a,b \geq 0;$$

(c) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}, \; a,b > 0;$$

(d) The logarithmic mean

$$L = L(a,b) := \begin{cases} a & if \quad a=b \\ \frac{b-a}{\ln b - \ln a} & if \quad a \neq b \end{cases}, \quad a,b > 0;$$

(e) The identric mean

$$I = I(a,b) := \begin{cases} a & if \quad a=b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & if \quad a \neq b \end{cases}, \quad a,b > 0;$$

(f) The p−logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} \quad a \neq b \\ a & \text{if} \quad a=b \end{cases}, \quad p \in \mathbb{R}\setminus\{-1,0\}; \; a,b > 0.$$

It is also known that L_p is monotonically nondecreasing in $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

Now, using the results of Section 2, some new inequalities are derived for the above means.

Proposition 5.1. *Let $p \geq 2$ and $0 < a < b$. Then we have the inequality:*

$$\left|L_p^p(a,b) - A\left(a^p, b^p\right)\right| \leq p\,(p-1)\,\frac{(b-a)^2}{12} A\left(a^{p-2}, b^{p-2}\right).$$

Proof. The assertion follows from (6) applied to $f(x) = x^p$, $x \in [a,b]$ .We omit the details. □

Proposition 5.2. *Let $0 < a < b$. Then we have the inequality:*

$$\left|L^{-1}(a,b) - \frac{A(a,b)}{G^2(a,b)}\right| \leq \frac{(b-a)^2}{6} A\left(a^{-3}, b^{-3}\right).$$

Proof. The assertion follows from (6) applied to $f(x) = \frac{1}{x}$, $x \in [a,b]$. We omit the details. □

Proposition 5.3. *Let $q > 1$ and $0 < a < b$. Then we have the inequality:*

$$\left|\ln I(a,b) - \ln G(a,b)\right| \leq \frac{(b-a)^2 B^{\frac{1}{p}}(p+1,p+1)}{2}\left[A\left(a^{-2q}, b^{-2q}\right)\right]^{1/q}.$$

Proof. The assertion follows from (8) applied to $f(x) = -\ln x$, $x \in [a,b]$. □

Proposition 5.4. *Let $p \geq 2$ and $0 < a < b$. Then we have the inequality:*

$$\left|L_p^p(a,b) - \frac{1}{2}\left[A^p(a,b) + A^p(a,b)\right]\right| \leq p\,(p-1)\,\frac{(b-a)^2}{96}\left[2A\left(a^{p-2}, b^{p-2}\right) + A^{p-2}(a,b)\right].$$

Proof. The assertion follows from (7) applied to $f(x) = x^p$, $x \in [a,b]$. □

Proposition 5.5. *Let $p > 1$ and $0 < a < b$. Then we have the inequality:*

$$\left|L^{-1}(a,b) - \frac{1}{2}\left[H^{-1}(a,b) + A^{-1}(a,b)\right]\right| \leq \frac{(b-a)^2}{48}\left[2A\left(a^{-3}, b^{-3}\right) + A^{-3}(a,b)\right].$$

Proof. The assertion follows from (7) applied to $f(x) = \frac{1}{x}$, $x \in [a,b]$ .. □

6. Applications for Composite Quadrature Formula

Let d be a division $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ of the interval $[a,b]$ and $\xi = (\xi_0, ..., \xi_{n-1})$ a sequence of intermediate points, $\xi_i \in [x_i, x_{i+1}]$, $i = 0, 1, 2, ..., n-1$. Then the following result holds:

Theorem 6.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L_1[a,b]$ where $a, b \in I$, $a < b$. If $\left|f''\right|$ is convex on $[a,b]$, then we have*

$$\int_a^b f(u)du = A(f,d,\xi) + R(f,d,\xi)$$

where

$$A(f,d,\xi) := \sum_{i=0}^{n-1} \frac{h_i}{2}\left[f(\xi_i) + (\xi_i - x_i)f(x_i) + (x_{i+1} - \xi_i)f(x_{i+1})\right]$$

The remainder $R(f,d,\xi)$ satisfies the estimation:

$$\begin{aligned} \left|R(f,d,\xi)\right| \leq\ & \frac{1}{24}\sum_{i=0}^{n-1}\left[(x_{i+1} - \xi_i)^3 + (\xi_i - x_i)^3\right]\left|f''(\xi_i)\right| \\ & + \frac{1}{24}\sum_{i=0}^{n-1}\left[(\xi_i - x_i)^3\left|f''(x_i)\right| + (x_{i+1} - \xi_i)^3\left|f''(x_{i+1})\right|\right] \end{aligned} \tag{16}$$

for any choice ξ of the intermediate points.

Proof. Apply Theorem 2.2 on the interval $[x_i, x_{i+1}]$, $i = 0, 1, 2, ..., n-1$ to get

$$\begin{aligned}
&\left| \int_a^b f(u)du - \frac{h_i}{2}\left[f(\xi_i) + (\xi_i - x_i) f(x_i) + (x_{i+1} - \xi_i) f(x_{i+1}) \right] \right| \\
\leq\ & \frac{1}{24}\left[(x_{i+1} - \xi_i)^3 + (\xi_i - x_i)^3 \right] \left| f''(\xi_i) \right| \\
&+ \frac{1}{24}\left[(\xi_i - x_i)^3 \left| f''(x_i) \right| + (x_{i+1} - \xi_i)^3 \left| f''(x_{i+1}) \right| \right].
\end{aligned}$$

□

Summing the above inequalities over i from 0 to $n-1$ and using the generalized triangle inequality, we get the desired estimation (16).

Corollary 6.2. *The following midpoint rule holds:*

$$\int_a^b f(u)du = M(f, d) + R_M(f, d)$$

where

$$M(f, d) := \sum_{i=0}^{n-1} h_i \left[f\left(\frac{x_i + x_{i+1}}{2} \right) \right]$$

and the remainder term $R_M(f, d)$ satisfies the estimation,

$$R_M(f, d) \leq \sum_{i=0}^{n-1} \frac{(h_i)^2}{24} \left(\left| f''(x_i) \right| + \left| f''(x_{i+1}) \right| \right).$$

Corollary 6.3. *The following perturbed trapezoid rule holds:*

$$\int_a^b f(u)du = T(f, d) + R_T(f, d)$$

where

$$T(f, d) := \sum_{i=0}^{n-1} \frac{h_i}{4}\left[f(x_i) + f(x_{i+1}) \right] - \sum_{i=0}^{n-1} \frac{h_i}{2} f\left(\frac{x_i + x_{i+1}}{2} \right)$$

and the remainder term $R_T(f, d)$ satisfies the estimation,

$$R_T(f, d) \leq \sum_{i=0}^{n-1} \frac{(h_i)^2}{96} \left(\left| f''(x_i) \right| + \left| f''\left(\frac{x_i + x_{i+1}}{2} \right) \right| + \left| f''(x_{i+1}) \right| \right).$$

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